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Second order hold and taylor series based discretization of SISO input time-delay systems †

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Abstract

Second order hold is a method that enables the discretization of input-driven nonlinear systems to be carried out with high precision. A new discretization scheme combining second order hold with the Taylor-series is proposed. The sampled-data representation and mathematical structure are explored. Both exact and approximate sampled-data representations are described in detail. The performance of the proposed algorithm is evaluated for three different systems. Various sampling rates, delay times and truncation orders of the Taylor-series are considered to investigate the proposed method. The results demonstrate that the proposed scheme is practical and easy to use for time-delay systems. Comparisons between the second, first and zero orders are given to show the advantages of the proposed method.

Keywords: Nonlinear system; Second order hold; Taylor series; Input delay; Time discretization

1. Introduction

In many physical, industrial and engineering systems, delays occur due to the finite capabilities of information processing and data transmission among the various parts of the system. In all of these cases, the time-delay factors have counteracting effects on the system behavior and usually lead to poor performance. Therefore, the subject of time-delay systems (TDS) has been investigated in the form of functional differential equations over the past three decades [1, 2].

Control systems with time delays exhibit complex behaviors. It is therefore difficult to apply the controller design techniques that have been developed during the last several decades to systems with any time delays in the variables. Thus, new design methods have been reported for time-delay control systems [3, 4].

Most of the approaches proposed so far deal with linear time-delay control systems and, in particular, with the stability analysis and behavior of such systems with constant and/or uncertain time delays [5-10]. Quite recently, nonlinear controllers were systematically synthesized for multivariable nonlinear systems in the presence of sensor and actuator deadtime [11].

The proposed discretization scheme is based on the Taylor series and uses a similar mathematical framework, which was previously developed for delay-free nonlinear systems [12, 13]. However, it should be mentioned that conventional numerical techniques, such as the Euler and Runge-Kutta methods, have been employed in order to obtain a sampled-data representation of the original continuous-time delay-free system [14, 15].

All of these approaches require a "small" time step in order to improve the precision; however, this may not be the case in control applications where large sampling periods are inevitably introduced due to

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physical and technical limitations [16-20].

In large sampling period systems, the Taylor series method was used to improve the performance of the controller [14, 21]. However, in previous papers, the zero-order hold (ZOH) and first-order hold (FOH) assumptions were used in the discretization method. The performance of the assumption is highly dependent on the input signal, and the sampling period should be short enough for the desired control precision.

A high-order method is one that provides increased accuracy with only a modest increase in the computational cost [22-24]. The ZOH and FOH assumptions no longer assure good control performance when large sampling intervals are adopted. Therefore, the second-order hold (SOH) assumption is introduced in this paper to enhance the performance in situations where a large sampling interval is inevitable.

In particular, the present study aims to develop a new method for the time discretization of SISO (single input and single output) nonlinear input-driven dynamic systems with time delay, based on the Taylor series and second-order hold assumption. This kind of discretization method inherits some of the system theoretic properties of the original continuous-time system (such as its equilibrium and stability properties). More importantly, however, it is a finite dimensional representation, which allows the direct application of existing nonlinear control system design techniques. Second, the Taylor-SOH discretization algorithm provides high precision under the sampling period restriction, as is confirmed by several illustrative case studies.

The paper is organized as follows: Section 2 presents mathematical preliminaries and Section 3 reviews recent techniques for the time discretization of delay-free nonlinear systems. Section 4 briefly presents the available time-discretization methods for linear time delay systems, and Section 5 includes the main results of this paper, in which the time-discretization method for nonlinear systems with time delay is introduced. Finally, several illustrative cases are presented in Section 6 demonstrating the effectiveness of the proposed discretization scheme. Section 7 provides a few concluding remarks drawn from this study.

2. Preliminaries

In the present study single-input nonlinear continuous-time control systems are considered with a state-space representation of the form:

$$\frac{dx(t)}{dt} = f(x(t)) + g(x(t))u(t-D) \tag{1}$$

where $x \in X \subset R^n$ is the vector of states and an open and connected set, $u \in R$ is the input variable and D is the system's constant time-delay (dead-time) that directly affects the input. It is assumed that $f(x)$ and $g(x)$ are real analytic vector fields on X.

An equidistant grid on the time axis with mesh $T = t_{k+1} - t_k > 0$ is considered, where $[t_k, t_{k+1}) = [kT, (k+1)T)$ is the sampling interval, T is the sampling period. It is also assumed that system (1) is driven by an input that is piecewise quadratic over the sampling interval, i.e., the second-order hold (SOH) assumption holds true.

2.1 SOH for delay-free system

For the SOH, while $D = 0$, and $kT \le t < kT + T$,

$$u(t) = u(kT) + \frac{u(kT) - u[(k-1)T]}{T}(t - KT) + \frac{1}{2}\frac{u(kT) - 2u[(k-1)T] + u[(k-2)T]}{T^2}(t-kT)^2 \tag{2}$$

It can also be written as,

$$u(t) = u(k) + \frac{u(k) - u(k-1)}{T}(t - KT) + \frac{1}{2}\frac{u(k) - 2u(k-1) + u(k-2)}{T^2}(t-kT)^2 . \tag{3}$$

Furthermore, let

$$s(k) = \frac{u(k) - u(k-1)}{T}, \tag{4}$$

$$a(k) = \frac{s(k) - s(k-1)}{T} \tag{5}$$

where, $s(k)$ represents the derivation at the instant kT, and $a(k)$ represents the second order derivation at the instant kT. Equation (3) can be abbreviated to,

$$u(t) = u(k) + s(k)(t-kT) + \frac{1}{2}a(k)(t-kT)^2 \tag{6}$$

This compact form will be used in the following part of this paper.

2.2 SOH for time-delay system

The time-delay D can also be expressed as,

$$D=(q+\delta)T=qT+\gamma \tag{7}$$

where, $q\in\{0,1,2,...\}$, $\delta\in(0,1)$ and $0<\gamma<T$. From (7), we can get

$$\gamma=\delta T . \tag{8}$$

Equivalently, the time-delay D is customarily represented as an integer multiple of the sampling period plus a time interval γ , such that γ is less than the sampling period [14, 15]. Based on the SOH assumption and the above notation, expressions of the SOH can be derived for time-delay systems step by step. Because of the existence of γ , in the procedure of the deductive method, it should be divided into two time intervals within the given sampling period:

$$\mathrm{I}_1=[kT,kT+\gamma), \tag{9}$$

$$\mathrm{I}_2=[kT+\gamma,kT+T). \tag{10}$$

First, while $q=0$, $\gamma\neq 0$,

$$u(t-D)\big|_{t\in I_1}=u(k-1)+s(k-1)[t-D-(k-1)T]$$
$$+\frac{1}{2}a(k-1)[t-D-(k-1)T]^2 , \tag{11}$$

$$u(t-D)\big|_{t\in I_2}=u(k)+s(k)(t-D-kT)$$
$$+\frac{1}{2}a(k)(t-D-kT)^2 . \tag{12}$$

Second, while $q=1$, $\gamma\neq 0$,

$$u(t-D)\big|_{t\in I_1}=u(k-2)+s(k-2)[t-D-(k-2)T]$$
$$+\frac{1}{2}a(k-2)[t-D-(k-2)T]^2 , \tag{13}$$

$$u(t-D)\big|_{t\in I_2}=u(k-1)+s(k-1)[t-D-(k-1)T]$$
$$+\frac{1}{2}a(k-1)[t-D-(k-1)T]^2 . \tag{14}$$

Therefore, the ordinary expression while $q\neq 0$ and $\gamma\neq 0$ is obtained conveniently as follows:

$$u(t-D)\big|_{t\in I_1}$$
$$=u(k-q-1)+s(k-q-1)[t-D-(k-q-1)T]$$
$$+\frac{1}{2}a(k-q-1)[t-D-(k-q-1)T]^2 , \tag{15}$$

$$u(t-D)\big|_{t\in I_2}$$
$$=u(k-q)+s(k-q)[t-D-(k-q)T]$$
$$+\frac{1}{2}a(k-q)[t-D-(k-q)T]^2 . \tag{16}$$

Let,

$$\Lambda_1(t)$$
$$=u(k-q-1)+s(k-q-1)[t-D-(k-q-1)T]$$
$$+\frac{1}{2}a(k-q-1)[t-D-(k-q-1)T]^2 , \tag{17}$$

$$\Lambda_2(t)$$
$$=u(k-q)+s(k-q)[t-D-(k-q)T]$$
$$+\frac{1}{2}a(k-q)[t-D-(k-q)T]^2 . \tag{18}$$

Therefore, (15) and (16) can be rewritten as follows:

$$u(t-D)=\begin{cases}\Lambda_1(t); & \mathrm{t}\in I_1\\ \Lambda_2(t); & \mathrm{t}\in I_2.\end{cases} \tag{19}$$

While $q\neq 0$ and $\gamma=0$, which means that the time delay D is an exact integer multiple of the sampling period *T*. In this situation, while $t\in[kT,kT+T)$, the expression should be

$$u(t-D)=u(k-q)+s(k-q)[t-D-(k-q)T]$$
$$+\frac{1}{2}a(k-q)[t-D-(k-q)T]^2 . \tag{20}$$

Let,

$$\Lambda(t)=u(k-q)+s(k-q)[t-D-(k-q)T]$$
$$+\frac{1}{2}a(k-q)[t-D-(k-q)T]^2 . \tag{21}$$

So that (20) can be rewritten as,

$$u(t-D)=\Lambda(t) . \tag{22}$$

At this point, it would be methodologically appropriate to succinctly review and delineate the time-discretization method available for delay-free ($D = 0$) nonlinear control systems that is based on the Taylor series and reported in [25]. The ensuing brief description of the Taylor discretization method for delay-free nonlinear systems will serve as a natural point of departure for the development of a similar in spirit discretization scheme that explicitly takes into account the presence of time-delay in the input variable ($D \neq 0$).

3. Time-discretization of delay-free nonlinear control systems

Initially, delay-free ($D = 0$) nonlinear control systems are considered with a state-space representation of the form,

$$\frac{dx(t)}{dt} = f(x(t)) + g(x(t))u(t) . \tag{23}$$

Under the SOH assumption and within the sampling interval, the solution of (23) is expanded in a uniformly convergent Taylor series [26], and the resulting coefficients can be easily computed by taking successive partial derivatives of the right hand-side of (23):

$$x(k+1) = x(k) + \sum_{\ell=1}^{\infty} \frac{T^{\ell}}{\ell!} \frac{d^{\ell} x}{dt^{\ell}}\bigg|_{t_k} = x(k) + \sum_{\ell=1}^{\infty} A^{[\ell]}(x(k),u(k)) \frac{T^{\ell}}{\ell!} , \tag{24}$$

where, $x(k)$ is the value of the state vector x at the instant $t = t_k = kT$ and $A^{[\ell]}(x,u)$ are determined recursively by:

$$A^{[1]}(x,u) = f(x) + ug(x),$$
$$A^{[\ell+1]}(x,u) = \frac{\partial A^{[\ell]}(x,u)}{\partial x}(f(x) + ug(x)) + \frac{\partial A^{[\ell]}(x,u)}{\partial u}\dot{u}, \tag{25}$$

where, $\ell = 1,2,3...$, $\dot{u} = du/dt$.

Therefore, an exact sampled-data representation (ESDR) of (23) can be derived by retaining the full infinite series of (9),

$$x(k+1) = \Phi_T(x(k),u(k)) = x(k) + \sum_{\ell=1}^{\infty} A^{[\ell]}(x(k),u(k)) \frac{T^{\ell}}{\ell!} . \tag{26}$$

Simultaneously, an approximate sampled-data representation (ASDR) of equation (23) is obtained by the truncation of the Taylor series of order N,

$$x(k+1) = \Phi_T^N(x(k),u(k)) = x(k) + \sum_{\ell=1}^{N} A^{[\ell]}(x(k),u(k)) \frac{T^{\ell}}{\ell!} , \tag{27}$$

where, the subscript T of the mapping Φ_T^N denotes the dependence on the sampling period T, and the superscript N denotes the finite series truncation order associated with the ASDR of equation (27).

4. Time-discretization of linear control systems with time-delay

It is now feasible to extend the aforementioned Taylor discretization method to nonlinear continuous-time systems with a constant time-delay ($D \neq 0$) in the input. In order to motivate the development of the proposed discretization procedure and draw the appropriate analogies from the field of linear systems, let us first begin the exposition of the paper's main results by briefly reviewing the ones available in the case of linear systems,

$$\frac{dx(t)}{dt} = Ax(t) + bu(t-D) , \tag{28}$$

where, A and b are constant matrices of appropriate dimensions. It is known that for any time interval $I = [t_i, t_f)$, the following formula holds true,

$$x(t_f) = e^{A(t_f - t_i)} x(t_i) + \int_{t_i}^{t_f} e^{A(t_f - \tau)} bu(\tau) d\tau . \tag{29}$$

As shown in (16) to (19), under the SOH assumption, the input variable expressions are different within the two subintervals $[kT, kT+\gamma)$ and $[kT+\gamma, kT+T)$. By successively applying formula (29), we readily obtain,

$$x(kT+\gamma) = e^{A\gamma}x(kT) + \int_{kT}^{kT+\gamma} e^{A(kT+\gamma-\tau)} b\Lambda_1(\tau)d\tau, \tag{30}$$

$$x(kT+T) = e^{A(T-\gamma)}x(kT+\gamma) + \int_{kT+\gamma}^{kT+T} e^{A(kT+T-\tau)} b\Lambda_2(\tau)d\tau, \tag{31}$$

where $\Lambda_1(\tau)$ and $\Lambda_2(\tau)$ are defined by Eqs. (17) and (18), respectively.

In light of Eq. (30), Eq. (31) yields,

$$x(kT+T) = e^{A(T-\gamma)}e^{A\gamma}x(kT) + \int_{kT+\gamma}^{kT+T} e^{A(kT+T-\tau)} b\Lambda_2(\tau)d\tau + e^{A(T-\gamma)}\int_{kT}^{kT+\gamma} e^{A(kT+\gamma-\tau)} b\Lambda_1(\tau)d\tau . \tag{32}$$

Let $\tau = kT - T + \gamma + \tau'$, then

$$e^{A(T-\gamma)}\int_{kT}^{kT+\gamma} e^{A(kT+\gamma-\tau)} b\Lambda_1(\tau)d\tau = \int_{T-\gamma}^{T} e^{A(2T-\gamma-\tau')} b\Lambda_1((k-1)T+\gamma+\tau')d\tau' = \int_{T-\gamma}^{T} e^{A(2T-\gamma-\tau)} b\Lambda_1((k-1)T+\gamma+\tau)d\tau . \tag{33}$$

Let $\tau = kT + \gamma + \tau'$, then

$$\int_{kT+\gamma}^{kT+T} e^{A(kT+T-\tau)} b\Lambda_2(\tau)d\tau = \int_{0}^{T-\gamma} e^{A(T-\gamma-\tau')} b\Lambda_2(kT+\gamma+\tau')d\tau' = \int_{0}^{T-\gamma} e^{A(T-\gamma-\tau)} b\Lambda_2(kT+\gamma+\tau)d\tau . \tag{34}$$

Therefore, Eq. (32) can be rewritten as,

$$x(kT+T) = e^{AT}x(kT) + \int_{0}^{T-\gamma} e^{A(T-\gamma-\tau)} bu_2(kT+\gamma+\tau)d\tau + \int_{T-\gamma}^{T} e^{A(2T-\gamma-\tau)} bu_1((k-1)T+\gamma+\tau)d\tau = e^{AT}x(kT) + \Gamma_0 + \Gamma_1 , \tag{35}$$

where,

$$\Gamma_0 = \int_{0}^{T-\gamma} e^{A(T-\gamma-\tau)} b\Lambda_2(kT+\gamma+\tau)d\tau,$$

$$\Gamma_1 = \int_{T-\gamma}^{T} e^{A(2T-\gamma-\tau)} b\Lambda_1((k-1)T+\gamma+\tau)d\tau.$$

Notice that the value of the state vector at $(k+1)T$ is defined by the states evaluated at kT and the two subinterval expressions, which can be obtained from the time-delay D and Eq. (19).

5. Time-discretization of nonlinear control systems with time-delay

Motivated by the linear approach described in section 4, a similar line of thinking is adopted for the nonlinear case as well. Indeed, by applying the Taylor series discretization method for nonlinear systems presented above to the $[kT, kT+\gamma)$ subinterval one immediately obtains the state vector evaluated at $kT+\gamma$,

$$x(kT+\gamma) = \Phi_\gamma(x(kT), \Lambda_1(kT)) , \tag{36}$$

where, the map Φ_γ can be derived through the direct application of formula (24), and the subsequent calculation of the corresponding Taylor coefficients can be realized through the recursive formulas (25). $x(kT)$ and $\Lambda_1(kT)$ are the instantaneous state vector and input value, respectively, at the instant kT. Furthermore, it can be derived from (17) that,

$$\Lambda_1(kT) = u(k-q-1) + s(k-q-1)(T-\gamma) + \frac{1}{2}a(k-q-1)(T-\gamma)^2 . \tag{37}$$

Similarly, the Taylor discretization method applied to the $[kT+\gamma, kT+T)$ subinterval yields the state vector evaluated at $(k+1)T$ as a function of $x(kT+\gamma)$ and the input value at the instant $kT+\gamma$,

$$x(kT+T) = \Phi_{T-\gamma}(x(kT+\gamma), \Lambda_2(kT+\gamma)) , \tag{38}$$

and,

$$\Lambda_2(kT+\gamma) = u(k-q) . \tag{39}$$

Based on (26), the above Eqs. (36) and (38) can be rewritten as follows:

$$x(kT+\gamma) = x(kT) + \sum_{\ell=1}^{\infty} A^{[\ell]}(x(kT), \Lambda_1(kT))\frac{\gamma^\ell}{\ell!} , \tag{40}$$

$$x(kT+T) = x(kT+\gamma)$$

$$+\sum_{\ell=1}^{\infty} A^{[\ell]}(x(kT+\gamma),\Lambda_2(kT+\gamma))\frac{(T-\gamma)^{\ell}}{\ell!}. \quad (41)$$

Furthermore, according to (27), the approximate sampled-data representations (ASDRs) of Eqs. (40) and (41) are obtained from the truncation of the Taylor series order of *N*, as shown below.

$$\begin{aligned} x(kT+\gamma) &= \Phi_{\gamma}^{N}(x(kT),\Lambda_1(kT)) \\ &= x(kT)+\sum_{\ell=1}^{N} A^{[\ell]}(x(kT),\Lambda_1(kT))\frac{\gamma^{\ell}}{\ell!}, \end{aligned} \quad (42)$$

$$\begin{aligned} &x(kT+T) \\ &= \Phi_{T-\gamma}^{N}(x(kT+\gamma),\Lambda_2(kT+\gamma)) \\ &= x(kT+\gamma) \\ &+\sum_{\ell=1}^{N} A^{[\ell]}(x(kT+\gamma),\Lambda_2(kT+\gamma))\frac{(T-\gamma)^{\ell}}{\ell!}. \end{aligned} \quad (43)$$

It should be emphasized that the functional representation of the $A^{[\ell]}$-coefficients of the map $\Phi_{T-\gamma}$ remains exactly the same subpart as that for the subinterval $[kT, kT+\gamma)$, and it is only necessary to reuse the same part with the aid of a symbolic software package such as MAPLE.

For the consecutive subintervals, by combining Eqs. (36) and (38), the desired sampled-data representation of the original system (1) is obtained.

$$\begin{aligned} &x(kT+T) \\ &= \Phi_{T}^{D}(x(kT),\Lambda_1(kT),\Lambda_2(kT+\gamma)) \\ &= \Phi_{T-\gamma}(\Phi_{\gamma}(x(kT),\Lambda_1(kT)),\Lambda_2(kT+\gamma)). \end{aligned} \quad (44)$$

Notice that a finite series truncation order *N* for the above series would naturally produce an ASDR:

$$x(kT+T) = \Phi_{T}^{N,D}(x(kT),\Lambda_1(kT),\Lambda_2(kT+\gamma)). \quad (45)$$

It can also be written as

$$x(k+1) = \Phi_{T}^{N,D}(x(k),\Lambda_1(k),\Lambda_2(k+\delta)). \quad (46)$$

Remark 1:

In the linear case, it is quite straightforward to show that the above formula (45) naturally reproduces result (35) as would be intuitively expected. Therefore, formula (45) represents its nonlinear analogue.

Remark 2:

The special case where $\gamma = 0$ and $D = qT$ frequently occur in practice when modeling and designing digital control systems. In this case, one easily obtains

$$\begin{aligned} x(k+1) &= \Phi_{T}^{D}(x(k),\Lambda(k)) \\ &= \Phi_{T}^{D}(x(k),u(k-q)) = \Phi_{T}(x(k),u(k-q)), \end{aligned} \quad (47)$$

as an ESDR, or

$$\begin{aligned} &x(k+1) \\ &= \Phi_{T}^{N,D}(x(k),u(k-q)) = \Phi_{T}^{N}(x(k),u(k-q)), \end{aligned} \quad (48)$$

as an ASDR of finite series truncation order *N*.

6. Case studies

The proposed time-discretization method for nonlinear control systems with time-delay using the Taylor series and SOH assumption is evaluated by applying it to three typical systems: a simple first order process system, a simple analytic second order system, and a two degrees of freedom mechanical system.

Different sampling periods and input delays were introduced in the simulation. At the same time, the MATLAB ODE solver was used to obtain the exact solutions in order to evaluate the proposed time-discretization method. The values obtained by the proposed method were compared with the results given by MATLAB. The partial derivative terms involved in the Taylor series expansion were determined recursively by MAPLE.

6.1 A process system

Let us start from a simple first order system. A chemical process system is considered in the simulation which is exactly the same as the system used in [14]. The system can be described as follows:

$$\frac{dx}{dt} = f(x)+g(x)u = -(1+2a)x+au-ux-ax^2. \quad (49)$$

In the simulation, *a=0.3* is used. The initial system state was assumed to be $x(0) = 0$.

Within the sampling interval, the solution of (49) is obtained by using a uniformly convergent Taylor series. According to the methodology described in the earlier sections, the sampled-data representation of the system is expressed as (42) and (43).

In this system,

$$f(x) = -(1+2a)x - ax^2,$$
$$g(x) = (a-x). \quad (50)$$

So that, the partial derivative terms $A^{[\ell]}(x,u)$ are determined recursively by (25).

The following sine-wave input is applied to the system:

$$u(t) = 0.8\sin(1.6\pi t) . \quad (51)$$

Therefore, the time-delay input applied to the system is as follows:

$$u(t-D) = 0.8\sin(1.6\pi(t-D)) . \quad (52)$$

Different sampling rates, time-delays and truncation orders of the Taylor-series are studied. Simultaneously, MATLAB 7.0 is used to calculate the accurate value. Two different situations are shown as follows.

Situation 1:

When the truncation order *N=3*, sampling time *T=0.09s*, and time delay *D=0.05s*, the state response for the SOH is shown in Fig. 1.

At the same time, the FOH and ZOH are used to provide a comparison with the SOH. Since MATLAB is used to calculate the exact values, a comparison of the response errors is shown in Fig. 2. It is obvious that the maximum error of 0.0104992 for the ZOH is decreased by 85.13% to 0.0015609 when the FOH is used, and that the maximum error of 0.0015609 for the FOH is decreased by 12.05% to 0.0013729 when the SOH is used.

Situation 2:

When the truncation order *N=4*, sampling time *T=0.2s*, and time delay *D=0.09s*, the state response for the SOH is shown in Fig. .

At the same time, the FOH and ZOH are used to provide a comparison. A comparison of the response errors is shown in fig.4. It is obvious that the maximum error of 0.0204395 for the ZOH is decreased by 58.15% to 0.008554 when using the FOH, and that the maximum error of 0.008554 for the FOH is decreased by 6.49% to 0.00799868 when using the SOH.

6.2 A second order system

In the previous section the proposed method was evaluated for a nonlinear first order system with input time-delay. In this section a simple second order system is studied.

The system is modeled as follows:

$$\ddot{x} = \dot{x}(1-x^2) - 2x + 2u . \quad (53)$$

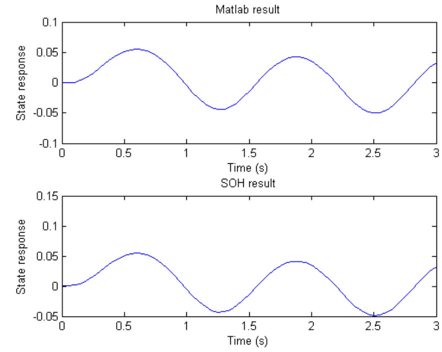

Fig. 1. State response (*N=3*).

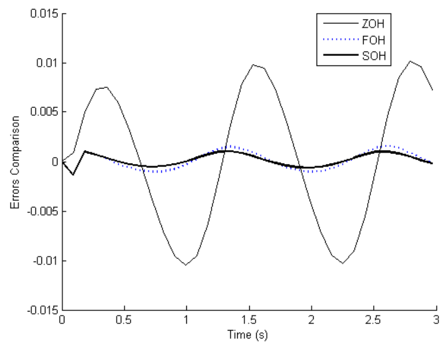

Fig. 2. Error comparison for situation *1*.

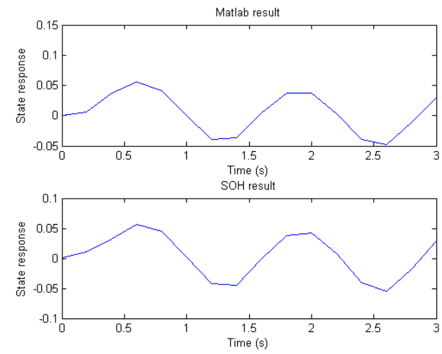

Fig. 3. State response (*N=4*).

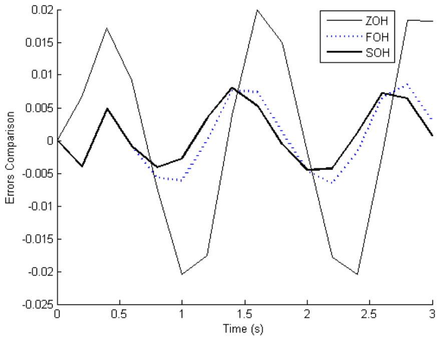

Fig. 4. Errors comparison for situation *2*.

It is assumed that the initial conditions are $x(0)=0.1$, $\dot{x}(0)=0$ and the input is an acceleration input,

$$u = 1/5*(t-D)\,\hat{}\,2\,. \tag{54}$$

The Taylor series discretization method requires a standard state-space representation form. The state variables of this system are defined as follows:

$$X_1 = x, \quad X_2 = \dot{x}\,. \tag{55}$$

Therefore, the state space system model of (53) is as follows:

$$\begin{aligned}\dot{X}_1 &= f_1(X)+g_1(X)u = X_2\\ \dot{X}_2 &= f_2(X)+g_2(X)u = X_2(1-X_1^2)-2X_1+2u\,.\end{aligned} \tag{56}$$

According to (42) and (43), the sampled-data representation of system (56) is obtained as follows:

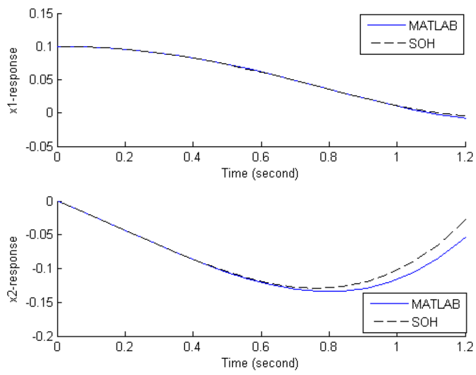

Fig. 5. State response of the system (*T=0.07s, D=0.035s*).

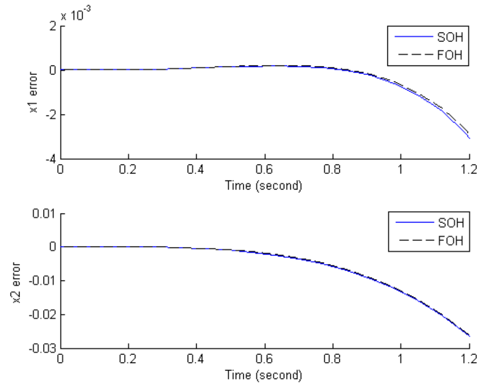

Fig. 6. Errors comparison (*T=0.07s, D=0.035s*).

$$\begin{aligned}&X_1(k+1)\\ &= X_1(k)+\sum_{\ell=1}^{N} A_1^{[\ell]}(X(k),\Lambda_1(kT))\frac{\gamma^\ell}{\ell!}+\sum_{\ell=1}^{N} A_1^{[\ell]}((X_1(k)\\ &+\sum_{j=1}^{N} A_1^{[j]}(X(k),\Lambda_1(kT))\frac{\gamma^j}{j!}),\Lambda_2(kT+\gamma))\frac{(T-\gamma)^\ell}{\ell!},\end{aligned} \tag{57}$$

$$\begin{aligned}&X_2(k+1)\\ &= X_2(k)+\sum_{\ell=1}^{N} A_2^{[\ell]}(X(k),\Lambda_1(kT))\frac{\gamma^\ell}{\ell!}+\sum_{\ell=1}^{N} A_2^{[\ell]}((X_2(k)\\ &+\sum_{j=1}^{N} A_2^{[j]}(X(k),\Lambda_1(kT))\frac{\gamma^j}{j!}),\Lambda_2(kT+\gamma))\frac{(T-\gamma)^\ell}{\ell!}\,.\end{aligned} \tag{58}$$

Where, the partial derivative terms $A^{[\ell]}(x,u)$ are determined, recursively, by (59) and (60), as follows:

$$\begin{aligned}&A_1^{[1]}(X,u) = f_1(X)+ug_1(X)\\ &A_1^{[\ell+1]}(X,u)\\ &= \frac{\partial A_1^{[\ell]}(X,u)}{\partial X_1} f_1 + \frac{\partial A_1^{[\ell]}(X,u)}{\partial X_2} f_2 + \frac{\partial A_1^{[\ell]}(x,u)}{\partial u}\dot{u},\end{aligned} \tag{59}$$

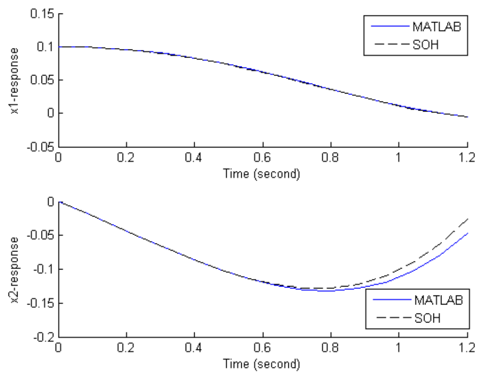

Fig. 7. State response of the system (*T=0.08s, D=0.03s*).

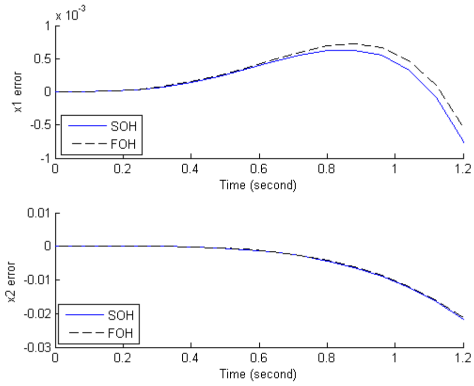

Fig. 8. Error comparison (*T=0.08s, D=0.03s*).

and

$$A_2^{[1]}(X,u) = f_2(X) + ug_2(X),$$
$$A_2^{[\ell+1]}(X,u) = \frac{\partial A_2^{[\ell]}(X,u)}{\partial X_1} f_1 + \frac{\partial A_2^{[\ell]}(X,u)}{\partial X_2} f_2 + \frac{\partial A_2^{[\ell]}(x,u)}{\partial u} \dot{u}. \quad (60)$$

Figs. 5 to 8 present the simulation results which enable the evaluation of the accuracy of the Taylor discretization method in this system. These numerical experiments were performed for a fixed truncation order, various input delays and various sampling periods. Throughout this case study, the truncation order was set to $N = 3$ for all simulations. Various sampling periods, T, 0.07 and 0.08, and several input time-delays, 0.035 and 0.03, are adopted in the simulations of this second order system.

Figs. 5 and 7 show a comparison between the results of the proposed Taylor-SOH method calculated by MAPLE and the MATLAB results. Figs. 6 and 8 show comparisons of the accuracy between the SOH and the FOH. From these figures, it can be seen that the sampling period significantly affects the accuracy of the proposed time-discretization method, as would be intuitively expected. It is also true that the SOH is better than the FOH under the simulation conditions.

6.3 A two degree-of-freedom mechanical system

In this section, a two-degree of freedom (DOF) mechanical system, which is composed of a slider, spring, damping components and a pendulum, is studied. A schematic of the system is shown in Fig. 9. The pendulum is hinged to a block mounted on a slider that is free to move on the guide. The motion of the slider is damped by springs, and the pendulum is damped by the rotational resistance in the hinge. The

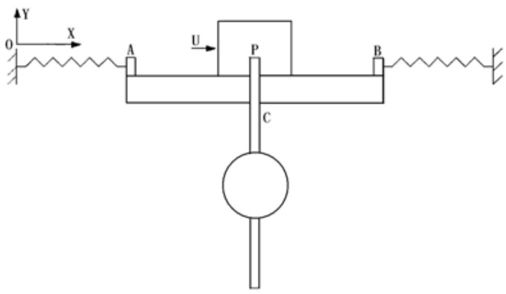

Fig. 9. Schematic diagram of pendulum system.

governing nonlinear differential equations of this system are obtained by using mechanical kinetics theory:

$$(m_1+m_2)\ddot{x} + m_1 l(\cos\theta + \mu\sin\theta)\ddot{\theta} + 2k(x-l_0) + m_1 l(\mu\cos\theta - \sin\theta)\dot{\theta}^2 - \mu(m_1+m_2)g = U,$$
$$m_1 l\cos\theta\ddot{x} + (I_c + m_1 l^2)\ddot{\theta} + m_1 g l\sin\theta = -M_0\dot{\theta}. \quad (61)$$

The state-space form of the system can be obtained from (61) by defining the state variables as follows:

$$X_1 = x, X_2 = \dot{x}, X_3 = \theta, X_4 = \dot{\theta}. \quad (62)$$

It gives

$$\dot{X}_1 = X_2, \quad \dot{X}_3 = X_4. \quad (63)$$

The resulting state-space equations are

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m_1+m_2 & 0 & m_1 l(\cos X_3 + \mu\sin X_3) \\ 0 & 0 & 1 & 0 \\ 0 & m_1 l\cos X_3 & 0 & I_c + m_1 l^2 \end{bmatrix} \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} X_2 \\ \begin{pmatrix} m_1 l(\sin X_3 - \mu\cos X_3)X_4^2 - \mu(m_1+m_2)g \\ -2k(X_1-l_0) + U \end{pmatrix} \\ X_4 \\ -M_0 X_4 - m_1 g l\sin X_3 \end{bmatrix}, \quad (64)$$

where, the parameters for this system are described in Table 1.

The initial states of the system are as follows:

$$X(0) = 0.04, \dot{X}(0) = 0, \theta(0) = 20^o, \dot{\theta}(0) = 0\ . \quad (65)$$

The sampled data representation of the 2-DOF mechanical system (64) is obtained by using the Taylor series discretization method:

$$X_i(k+1)$$
$$= X_i(k) + \sum_{\ell=1}^{N} A_i^{[\ell]}(X(k), \Lambda_1(kT))\frac{\gamma^\ell}{\ell!} + \sum_{\ell=1}^{N} A_i^{[\ell]}((X_i(k)$$
$$+\sum_{j=1}^{N} A_i^{[j]}(X(k), \Lambda_1(kT))\frac{\gamma^j}{j!}), \Lambda_2(kT+\gamma))\frac{(T-\gamma)^\ell}{\ell!}\ , \quad (66)$$

Table 1. Parameters of the pendulum system.

$m_1 = 0.654kg$	mass of slider
$k = 100N/m$	spring coefficient
$l_0 = 0.025m$	initial length of the spring
$I_c = 0.0014kg \cdot m^2$	inertia about the center
$\mu = 0.2$	coefficient of friction
$m_2 = 0.7925kg$	mass of pendulum
$l = 0.2m$	length of the rod
$g = 9.8m/s^2$	gravity
$M_0 = 0.2kg \cdot m \cdot \deg/s^2$	dry friction from the pendulum

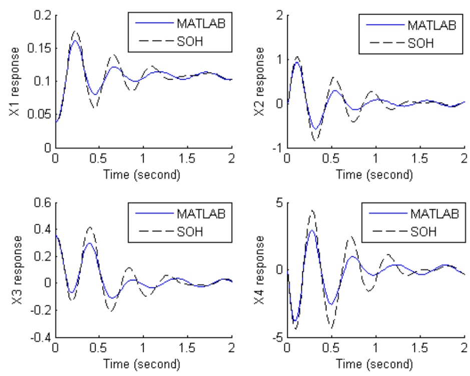

Fig. 10. State response (*N=1*).

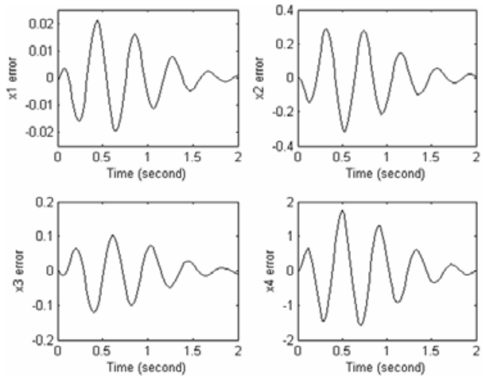

Fig. 11. Error of SOH method (*N=1*).

where *i=1,2,3,4* and the partial derivative terms $A_i^{[\ell]} = (x, u)$ are determined recursively by (25).

The input force acting on the system is assumed to be $U = 19.5 + 0.5\sin(4\pi(t-D))$. Suppose that the sampling rate for the system is limited to $50Hz$. In this case, the sampling period is not less than 0.02s. The input time-delay is D=0.015s. In such a situation, several simulations for different truncation orders N from 1 to 3 were performed in MAPLE with the proposed Taylor-SOH method, as shown in Figs. 10 to 15. Figs.10, 12 and 14 show the state responses for the different truncation orders, N. Figs.11, 13 and 15 show the errors of the Taylor-SOH method for the different truncation orders, N. The state X_1 is the position of the slider, X_2 is the velocity of the slider, X_3 is the angle of the pendulum from the reference and X_4 is the angular velocity of the pendulum.

From these figures, it can be seen that the accuracy was greatly improved when *N* was increased from *1* to *2*. The accuracy was increased by choosing a bigger value of *N*. Table 2 shows the maximum error of every state variable for the different truncation orders. From this table, it can be observed that when *N* is increased from *3* to *4*, no obvious improvement in the accuracy can be observed. Therefore, we should select an appropriate value of *N* under the limitations of the sampling time and the accuracy demanded.

This example of a higher order system clearly shows that the proposed Taylor-SOH method is able to discretize a nonlinear system with input time-delay quite accurately.

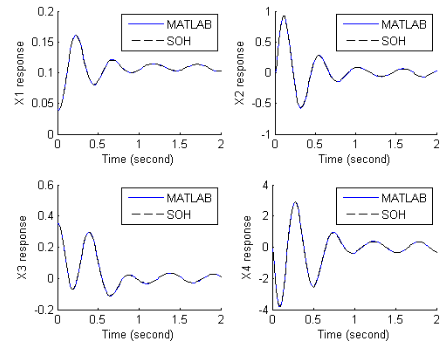

Fig. 12. State response (*N=2*).

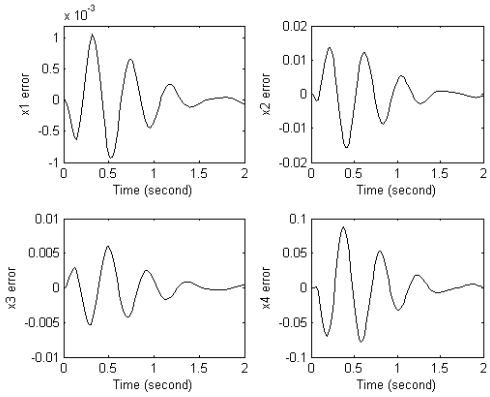

Fig. 13 Error of SOH method (*N=2*).
Table 2. Comparison of maximum errors for different truncation orders (*T=0.02s, D=0.015s*).

State	Max error of Taylor series-SOH assumption			
	N=1	N=2	N=3	N=4
X1	0.02110	0.001061	0.0000559	0.0000398
X2	0.31824	0.015593	0.0007465	0.0005809
X3	0.12019	0.005959	0.0002920	0.0002300
X4	1.75000	0.087588	0.0040231	0.0033427

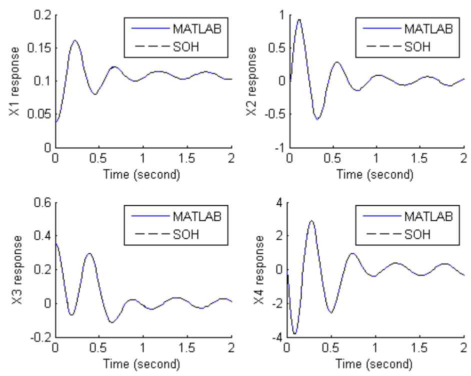

Fig. 14. State response (*N=3*).

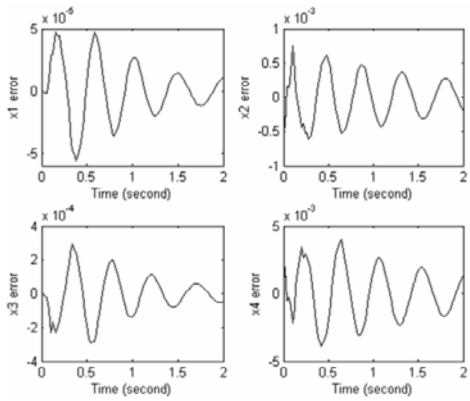

Fig. 15. Error of SOH method (*N=3*).

7. Conclusion

A scheme based on the Taylor series combined with the second-order hold assumption is proposed for the derivation of a discrete-time representation of a nonlinear control system with time-delay. The mathematical structure of the new discretization scheme is explored and characterized as useful for establishing the concrete connections between the numerical and system-theoretic properties. The derived time-discretization method provides a finite-dimensional representation for nonlinear control systems with time-delay, thereby enabling the application of existing nonlinear controller design techniques to such systems.

The performance of the proposed time-discretization procedure is evaluated by using three case studies with increasing complexity: a first order process control system, a second order system and a two degree-of-freedom mechanical system. Various sampling rates and time-delay values are considered in the example studies. The simulation results are compared with those given by MATLAB, in order to verify the accuracy of the proposed method. These examples demonstrate how to use the proposed method to solve a real system. In these cases, even when the sampling period is large with input time-delay, the Taylor series combined with the SOH can satisfy the accuracy requirement of the systems.

At the same time, some comparisons are made between the SOH with the FOH and ZOH methods when combined with the Taylor series in the discretization procedure. The results show that the SOH method is much better at retaining the high precision of the input signals than the FOH and ZOH methods, in such cases as sinusoidal and acceleration inputs.

Furthermore, general expressions for different order hold discretization schemes will be the subject of a future publication.

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